Note on Holographic RG Flow in String Cosmology

Miao Li[1] and Feng-Li Lin[2]

[1] *Institute of Theoretical Physics*
Academia Sinica
Beijing 100080
and
Department of Physics
National Taiwan University
Taipei 106, Taiwan
mli@phys.ntu.edu.tw

[2] *Department of Physics*
Tamkang University
Tamsui, Taipei 25137, Taiwan
linfl@mail.tku.edu.tw

Abstract

We propose a new holographic C-function for the accelerating universe defined in the stringy frame motivated mainly by the fact that the number of degrees of freedom should be infinite for a physically infinite universe. This is the generalization of Strominger's recent proposal of the holographic C-function to the asymptotically non-de Sitter universe. We find that the corresponding C-theorem holds true if the universe accelerates toward the weak coupling regime driven by the exponential dilaton potential. It also holds in other simple cases.

1 Introduction

Recent results in observational cosmology provide substantial evidence for the existence of dark energy [1]. String theory for the first time has to cope with a possible positive cosmological constant, or a tiny positive vacuum energy which may depend on some moduli. As pointed out in [3], with either a cosmological constant or the popular quintessence model, the universe's expansion will forever accelerate, resulting in a future horizon. This poses conceptual problems in string theory thus far formulated [2].

Despite the fact that no one has come up with a credible model with a positive vacuum energy in string theory (for some attempts, see [4]), people have speculated on the possible microscopic descriptions of such a universe. This includes a possible dS/CFT correspondence which generalizes the usual AdS/CFT correspondence, and a possible holographic horizon theory and matrix models [6]. The dS/CFT approach has since been followed up by many authors [7]. It appears to us that both lines of approach contain some ingredients of truth and may even be related to each other. Both are partially motivated by the conjecture that the number of degrees of freedom in a de Sitter universe is finite [8]. It is thus a good question to ask whether at a given time, the number of degrees of freedom observable to a given observer is finite and can be written as a local function of geometry at that given time. By analogy with the AdS/CFT correspondence, there is an answer if the asymptotic geometry in the future is de Sitter, as discussed in [9] and the second reference of [5].

The central charge, or more generally, the measure of the number of degrees of freedom defined in [9] for a geometry asymptotating de Sitter space is given by

$$C(t) = \frac{1}{H^{d-1}G}, \tag{1}$$

where H is the Hubble constant and G is the Newton constant in $d+1$ dimensions. For a de Sitter space, $H = 1/R$, R is the radius of the cosmic horizon, the formula (1) coincides with the one for the cosmic entropy. For a geometry asymptotically de Sitter, $C(t)$ is a function of time, and the formula can be inferred by comparing to the known formula in the AdS/CFT correspondence [10]. One must keep in mind that such formula can be related to the Weyl anomaly only when the boundary theory lives in an even dimension, that is, when d is even. For odd d, in particular for $d = 3$ of the observed universe, (1) stands as a good guess. Now just as in the AdS/CFT correspondence, Einstein equations together with the null energy condition ensure nondecrease of the central charge function.

When one claims that the central charge increases as the universe evolves, one is not claiming a law similar to the second law of thermodynamics, since in calculating $\frac{dC}{dt}$ one has not resorted to thermodynamics at all. Actually an arrow of time is already chosen by assuming the positivity of the Hubble constant H. In an odd dimensional universe, H has to be positive for C to be positive. In an even dimensional universe, H does not have to be positive in defining C, however in showing $\frac{dC}{dt} \geq 0$, one has to assume $H \geq 0$.

While (1) may well be a good candidate for the C-function in a universe asymptotically de Sitter, it may fail for more general situation when the fate of the universe is completely different. For instance, in a universe driven by quintessence in the future, the size of the horizon increases indefinitely, one does not expect the same formula to hold here. This is already the case in the AdS/CFT correspondence. One example is the gravity dual of noncommutative Super Yang-Mills, where the asymptotic geometry is not AdS, and naturally the C-function must be worked out separately as in [11]. In the present context, one may consider the case when the acceleration of the expansion vanishes, the borderline of the quintessence model. The radius factor $a(t)$ is proportional to t, so the physical size of the future horizon

$$R(t) = a(t) \int_t^\infty \frac{dt}{a(t)}, \tag{2}$$

diverges for any finite time t, thus one expects the measure of the number of degrees of freedom also diverge. Formula (1) however gives a finite number.

If the quintessence field is the dilaton, we shall see that the Hubble constant in the stringy frame actually vanishes if the string coupling is driven to the weak coupling regime, this motivates us to propose the definition

$$C(t) = \frac{1}{H_s^{d-1} G_s}. \tag{3}$$

Now the Newton constant in the stringy frame depends on the dilaton thus depends on time in general. The above formula is taken to be a good candidate only in the weakly coupled string theory. We shall show that for known quintessence potentials, a C-theorem is valid for the above definition. Actually, a C-theorem holds only when the universe expansion accelerates if the potential is an exponential of the dilaton. Thus, C-theorem formulated in terms of (3) is more restrictive than the one formulated in terms of (1). We take this as a good sign, namely not all solutions to the Einstein equations with a reasonable matter content are all plausible universes.

The strongest support to the formula (3) comes from the following fact. For an exponential quintessence potential for the dilaton ϕ, the solution $a(t)$ scale as $t^{1+\kappa}$. In this case, the future horizon size (2) is $(1/\kappa)t$. As $\kappa \to 0$, this blows up. The area of the future horizon then scales as $(1/\kappa^{d-1})t^{d-1}$. It happens that in this case the central charged defined in (3) also scale in the same way in terms of both κ and t.

If the universe is driven to the strongly coupled regime far into the future, one may either use (3) or a different definition motivated by M theory. If (3) is used, the c-theorem always holds. If a M theory definition is adopted, we shall see that a corresponding C-theorem also holds, and the condition is simply $p < \rho$. However, no natural definition will give an infinite C for the case $a(t) \sim t$. Does this impossibility imply that this case is impossible in the M theory regime, or alternatively the fate of the string coupling is zero in the far future?

Our discussions in the following will be focused in the far future, since there one is devoid of the problem of complicated matter as the dark energy dominates.

2 Holographic C-theorem of the FRW Universe in the Einstein frame

For more general consideration, let us start with the action of the dilatonic gravity in d+1+D dimensions for string cosmology [13]:

$$S = S_0 + S_U = -\int d^{d+1+D}X \, \sqrt{-G} e^{-2\phi}[R + 4(\nabla\phi)^2 - U(\phi)] \, , \tag{4}$$

where $U(\phi)$ is the dilaton potential which is zero at tree level for critical string(if $d+1+D = d_c$, the critical dimension) but can get nontrivial quantum corrections; and for the noncritical string $U = \frac{2}{3}(d_c - d - 1 - D)$ at tree level.

Compactifying the theory on D-dimensional torus T^D with the following metric ansatz

$$ds^2 = G_{MN}dX^M dX^N = \hat{g}_{\mu\nu}(x)dx^\mu dx^\nu + \sum_{i=1}^{D} e^{2\sigma_i(x)} dy_i^2 \, , \tag{5}$$

where $\hat{g}_{\mu\nu}$ is the metric of the (d+1)-dimensional noncompact space. After some manipulations, we arrive the following dimensionally-reduced action[1]:

$$S = -\int d^{d+1}x \, \sqrt{-\hat{g}} e^{-\Phi}[\hat{R} + (\hat{\nabla}\Phi)^2 - \sum_{i=1}^{D}(\hat{\nabla}\sigma_i)^2 - U(\Phi, \sigma_i)] \, , \tag{6}$$

where the hat quantities are with respect to metric $\hat{g}_{\mu\nu}$, and the new field $\Phi \equiv 2\phi - \sum_{i=1}^{D}\sigma_i$. Also note that the moduli σ_i could get a nontrivial potential from quantum correction.

We transform the action into the Einstein one by

$$g_{\mu\nu} = e^{\frac{-2\Phi}{(d-1)}} \hat{g}_{\mu\nu} \, , \tag{7}$$

and the action becomes

$$S = -\int d^{d+1}x \, \sqrt{-g}[R - \frac{1}{d-1}(\nabla\Phi)^2 - \sum_{i=1}^{D}(\nabla\sigma_i)^2 - V(\Phi, \sigma_i)] \, , \tag{8}$$

where $V(\Phi, \sigma_i) \equiv e^{\frac{2\Phi}{d-1}} U(\Phi, \sigma_i)$.

From this relation, a potential $U = \sum_{n=1} c_n g^{2(n-1)}$ calculated from perturbative string theory will be transformed into $g^{\frac{4}{d-1}} U$ in the Einstein frame, where n is the number of loop and the string coupling

$$g = e^{\Phi/2} \, . \tag{9}$$

Moreover, the 1-loop potential $V = e^{\frac{2}{d-1}\Phi}$ is dominant as $\Phi \to -\infty$ but is just marginally able to drive the universe in acceleration as remarked in [14]. In this note we assume that the expansion of the Universe is driven by a generic potential V in the late time so that we

[1] Set the volume of T^D parameterized by y_i equal to one.

need to require V to be a slowly varying *positive* function for our Universe to be de Sitter-like to conform with the astronomical observation in [1].

Note that both the "dilaton" Φ and the internal moduli σ_i become canonical scalars in the Einstein frame, however, a nontrivial potential mixes these scalars. Moreover, from the action we can derive the energy density and pressure if we treat the "matter" part as perfect homogeneous fluid, it turns out to be

$$\rho = \frac{1}{d-1}\dot{\Phi}^2 + \sum_{i=1}^{D} \dot{\sigma}_i^2 + V(\Phi, \sigma_i) \ , \tag{10}$$

$$p = \frac{1}{d-1}\dot{\Phi}^2 + \sum_{i=1}^{D} \dot{\sigma}_i^2 - V(\Phi, \sigma_i) \ , \tag{11}$$

where the dot is the derivative with respect to the time coordinate t of the FRW metric

$$ds^2_{FRW} = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + e^{2\lambda(t)}d\Sigma_k, \tag{12}$$

where $k = -1, 0, 1$ for open, flat and closed Universes.

The FRW equations are[2]

$$H^2 = \frac{-k}{a^2} + \frac{1}{d(d-1)}\rho \ , \tag{13}$$

$$\dot{H} = \frac{k}{a^2} - \frac{1}{2(d-1)}(p+\rho) \ , \tag{14}$$

and the field equations for the scalars are

$$\frac{2}{d-1}(\ddot{\Phi} + dH\dot{\Phi}) + \frac{\partial V}{\partial \Phi} = 0 \ , \tag{15}$$

$$2(\ddot{\sigma}_i + dH\dot{\sigma}_i) + \frac{\partial V}{\partial \sigma_i} = 0 \ . \tag{16}$$

where we have defined the Hubble constant $H = \dot{\lambda}$ according to the scale factor $a = e^{\lambda(t)}$.

From the fact of $p + \rho = \frac{2\dot{\Phi}^2}{d-1} > 0$ one can immediately see that H is monotonically decreasing for $k \leq 0$[3]. This is the key observation of C-theorem for the holographic dual CFT in [5] that the central function (1) is monotonically increasing during time evolution as long as ρ tends to constant in the future infinity such that the universe is asymptotically de Sitter, which is a UV fixed in the dual CFT picture. Although this coincidence strengthen the dS/CFT correspondence proposed in [9], it seems less powerful to constrain the possible cosmological scenarios since it is valid for very generic matters as long as the general positive theorem holds. This is in contrast to the Fischler-Susskind cosmic holographic conjecture [12] which gives constraints to the possible cosmological behavior.

[2] Here we set $16\pi G_N$ equal to one.

[3] We will restrict ourselves to k=0 case only in the following discussions.

Another puzzle regarding the holographic C-theorem is that the C-theorem is not time-reversal invariant but the field equations are. How can the second order field equations know the 2nd law of thermodynamics which is implied by the increase of the C-function as the Universe evolves, or the number of degrees of freedom in the dual CFT? The resolution to the puzzle is that we have required the Hubble constant to be positive, which is odd under time-reversal so that a time direction is picked up when we assume the expanding Universe. Especially for even d, to ensure the positivity of the C-function, the condition $H > 0$ should be required for the notion of C-function to be sensible.

3 Accelerating Universe

As known that the cosmological behavior is observer dependent, and thus frame dependent. Since we start with an effective dilatonic supergravity action, it also makes sense to consider the holographic dual picture in the stringy frame, then the stringy nature may yield the new feature in the holographic consideration. We will see that this is indeed the case and the holographic C-theorem may serve as a constraint to the possible cosmological scenarios.

Since the equations of motion alone are not enough to verify the C-theorem in stringy frame as we shall see later, we need to integrate out the equations of motion for specific checking. For simplicity, we freeze all the moduli but one which is defined as the length variable of the moduli space as $Z = \int dt \sqrt{\frac{1}{d-1}\dot{\Phi}^2 + \sum_{i=1}^{D} \dot{\sigma}_i^2}$ [14], then $\rho = \dot{Z}^2 + V(\Phi(Z), \sigma_i(Z))$ and the new equation of motion for Z by combining the ones for Φ and σ_i is

$$2(\ddot{Z} + dH\dot{Z}) + \frac{\partial V}{\partial Z} = 0 \ , \tag{17}$$

where we have used $\frac{\partial V}{\partial Z}\dot{Z} = \frac{\partial V}{\partial \Phi}\dot{\Phi} + \sum_{i=1}^{D} \frac{\partial V}{\partial \sigma_i}\dot{\sigma}_i$.

Using the equations of motion by starting with $\dot{\rho} = 2\dot{Z}\ddot{Z} + \frac{\partial V}{\partial Z}\dot{Z}$ and the "velocity" formula

$$\dot{Z} = \pm\sqrt{\rho - V} \ , \tag{18}$$

we can arrive

$$\rho = \mp 2\sqrt{\frac{d}{d-1}} \int dZ \ \sqrt{\rho(\rho - V)} \ . \tag{19}$$

The particle horizon and the scale factor can also be put in the integral form as

$$\begin{aligned} R_H &= \int \frac{dt}{a} = \pm \int dZ \frac{1}{a\sqrt{\rho - V}} \ , & (20) \\ a &= exp\{\pm\sqrt{\frac{1}{d(d-1)}} \int dZ \ \sqrt{\frac{\rho}{\rho - V}} \ \} \ . & (21) \end{aligned}$$

Qualitatively if energy density ρ is asymptotically dominated by the potential, then a will diverge more rapidly than $(\rho - V) \to 0$ in the late time such that the integral defining R_H is finite for all time, i.e. there exists the future horizon which naturally arises in de Sitter space and the accelerating Universe driven by quintessence as shown in [3]. On the other hand, the accelerating Universe requires

$$\frac{\ddot{a}}{a} = \frac{1}{d(d-1)}[-\dot{\Phi}^2 - (d-1)\sum_{i=1}^{D}\dot{\sigma}_i^2 + V(\Phi,\sigma_i)] > 0 \;, \tag{22}$$

which leads to a constraint on the Hubble constant

$$H^2 > \frac{1}{d-1}[\frac{\dot{\Phi}^2}{d-1} + \sum_{i=1}^{D}\dot{\sigma}_i^2] = \frac{1}{d-1}(\rho - V) \;, \tag{23}$$

where we have combined (13) and (22) together to arrive at (23). Moreover, using (13) and the relation $p = \rho - 2V$, then (23) is equivalent to the constraint on the equation of state

$$\omega \equiv \frac{p}{\rho} < \frac{2-d}{d} \;, \tag{24}$$

which is also known as the bound for the quintessence matters.

Specifically we can choose a simplest form of potential[4], the exponential potential $V \propto e^{\pm\alpha\Phi}$, with $\alpha \geq 0$. As shown in [16] there exists an attractive fixed point in the solution space such that ω is constant in time. If we choose the branch with $V \propto e^{-\alpha\Phi}$, we need to require that $\dot{\Phi} > 0$ to agree with the physical cosmology of decreasing not increasing dark vacuum energy, and from (10) we have

$$\dot{\Phi} = \sqrt{(d-1)(\rho - V)} \;. \tag{25}$$

This leads to that the string coupling $g = e^{\Phi/2}$ evolves toward the strong string coupling regime. We will refers to this as the *strong coupling branch*. The other branch is to choose $V \propto e^{\alpha\Phi}$ such that the energy density diminish asymptotically, so does the string coupling; we refer to this as the *weak coupling branch* with

$$\dot{\Phi} = -\sqrt{(d-1)(\rho - V)} \;. \tag{26}$$

From (19) we can relate α to ω by

$$\alpha = \frac{\sqrt{2d(1+\omega)}}{d-1} \tag{27}$$

for both the strong and weak coupling branches. Using (27) the accelerating Universe condition (24) can be translated into constraint on the decaying rate of the potential as

$$\alpha < \frac{2}{d-1} \tag{28}$$

[4]For the moment we will freeze the moduli σ_i for all i, therefore $Z = \Phi/\sqrt{d-1}$.

for the exponential potential case [16].

Another simple form of the quintessential potential is the power-law one given by[5]

$$
\begin{aligned}
\rho &= \rho_0(-Z)^{-\delta} , && (29)\\
\rho - V &= C_0(-Z)^{-\gamma} , && (30)
\end{aligned}
$$

where a, b are positive constants and we have chosen the weak coupling branch so that $\rho \to V$ as $Z \to -\infty$ and

$$
\dot{Z} = -\sqrt{\rho - V} . \tag{31}
$$

Note that in this case ρ/V is no longer a constant so that it leads to no fixed point in the solution space but to the tracker solution [15].

From (19) we can obtain the following relations

$$
\begin{aligned}
C_0 &= \frac{(d-1)\delta^2\rho_0}{4d} , && (32)\\
\gamma &= \delta + 2 . && (33)
\end{aligned}
$$

It is then easy to see that the Universe is accelerating for large enough Z for any a and ρ_0 which instead will be constrained by the condition of no additional long range force besides the known ones.

4 Holographic C Theorem in Stringy frame

As emphasized the cosmological behavior is observer and thus frame dependent. In the stringy frame the Hubble constant and the effective Newton constant are different from the ones in Einstein frame and are defined with respect to the stringy metric $\hat{g}_{\mu\nu}$ via (7) and (12), the results are

$$
\begin{aligned}
H_s &= g^{\frac{-2}{d-1}}(H + \frac{\dot{\Phi}}{d-1}) , && (34)\\
G_N^{(s)} &= g^2 G_N , && (35)
\end{aligned}
$$

where G_N and H are the Newton constant and Hubble constant respectively in the Einstein frame, and G_N is time-independent by definition. From these, the inverse C-function is

$$
H_s^{d-1} G_N^{(s)} = G_N(H + \frac{\dot{\Phi}}{d-1})^{d-1} . \tag{36}
$$

Note that although the scale behavior of H_s is different from the one of H due to the dressed string coupling factor $g^{\frac{-2}{d-1}}$, the resultant C-function has the same scale behavior as the one

[5]To distinguish from the results for the case of exponential potential we instead choose not Φ but the length variable Z as the only unfrozen moduli.

in the Einstein frame proposed by Strominger [9] but differs by a sub-leading term. This fact is essential for the C-theorem in the stringy frame to hold and be closely related to the C-theorem in the Einstein frame. For the special case of that H and $\dot{\Phi}$ have the same scale behavior, the resultant C-function differs from the one in the Einstein frame only by an overall coefficient which is irrelevant to the validity of the C-theorem as long as the coefficient is positive.

For the string cosmology to make sense it requires that the stringy Hubble constant is positive to agree with the expanding universe observation even for an observer in the stringy frame. This condition is also necessary for the holographic C-function in the stringy frame to be sensible when d is even. For the *strong coupling branch* this is always true since $H > 0$, $\dot{\Phi} > 0$. On the other hand, for the *weak coupling branch* $\dot{\Phi} < 0$, the condition for $H_s > 0$ means that

$$H^2 > \frac{\dot{\Phi}^2}{(d-1)^2} \, . \tag{37}$$

Surprisingly, this is coincident with the condition (23) in the Einstein frame for the accelerating Universe driven by dilaton alone but freezing other moduli[6]. *We conclude that the Universe in the weak string coupling branch is observed to be expanding but not contracting if it is accelerating in the Einstein frame.* We have no clue of physical reasons for this coincidence.

The other reason to justify our choice of C-function (3) as remarked in the Introduction is to look into the borderline case of $H_s = 0$. Its scale factor is linear in t, i.e. $a(t) = t$ so that the size of the future horizon (2) diverges but the inverse Hubble constant in Einstein frame $1/H$ is finite, so is the C-function (1). This is not compatible with the conventional wisdom of quantum theory that infinite physical universe should contain infinite number of degrees of freedom. Part of the reason for the above failure of (1) is that the asymptotic geometry of $H_s = 0$ universe is not de Sitter. It is then physically motivated to find the appropriate C-function for the asymptotically non-de Sitter universes, which are generic for the string cosmology with dilaton as the dominant driving force of the cosmological expansion in the late time. Since the driving force of the expansion of the universe is assumed to come from the stringy moduli, it is natural to use our C-function (3) defined in the stringy frame to incorporate the physical effect of dilaton. Moreover, (3) diverges at $H_s = 0$ and is compatible with the infinite size of the future horizon. Since the case of $H_s = 0$ is at the borderline of the accelerating Universe, it is then reasonable to postulate the stringy C-function (3) as the measure of the number of degrees of freedom for the accelerating universe driven by dilaton.

We then need to make sure the inverse C function (36) is monotonically decreasing such that Holographic C-theorem holds. It turns out we need to check if

$$\dot{H} - \frac{d}{d-1} H \dot{\Phi} - \frac{1}{2} \frac{\partial V}{\partial \Phi} < 0. \tag{38}$$

It is in general unable to determine the sign of the above expression by using the equations

[6] The moduli dynamics is in fact positive to the constraint (37).

of motion alone as in the case of Einstein frame, however, one can check it explicitly for specific potential form.

For the case of exponential potential, the explicit time dependence of stringy Hubble constant is

$$H_s = g^{\frac{-2}{d-1}} \left(1 \pm \sqrt{\frac{d(1+\omega)}{2}}\right) H \ , \tag{39}$$

and

$$H \propto \frac{1}{t} \ . \tag{40}$$

where the $+$ sign is for the strong coupling branch, and the $-$ sign for the weak one.

In the case when the dilaton is driven to $-\infty$, the central function assumes the form $C(t) = f t^{d-1}$, and $a(t)$ scales as

$$a(t) \sim t^{\frac{2}{d(1+\omega)}}, \tag{41}$$

The exponent is greater than 1, if the quintessence condition $1+\omega < 2/d$ is satisfied. As $1+\omega$ approaches $2/d$, the exponent in (41) approaches 1 and the future horizon size defined in (2) diverges as $(1/\kappa)t$, where $\kappa = 2/(d(1+\omega)) - 1$ approaching zero. Thus the area of the horizon diverges according to $(1/\kappa^{d-1})t^{d-1}$. Happily, the central charge defined in the string frame diverges in the exactly the same manner, namely $f \sim 1/\kappa^{d-1}$ for small κ.

As remarked before, here H_s is proportional to H so that the stringy C-theorem holds as long as $H_s > 0$ and the C-theorem in the Einstein frame holds. The latter is always true as long as positive energy theorem is assumed. So in the strong coupling branch, the C-theorem is true for any ω since $H_s > 0$ always. However, as the string coupling becomes large one can no longer trust the perturbative string picture, instead we expect that an M-theory cycle will open up and it is then more natural to use the $d+2$-dimensional dual weakly coupled theory picture to discuss the dynamics. We will switch to this picture in the next section. On the other hand, for the weak coupling branch the holographic C-theorem is valid only when $H_s > 0$. In this case, the C-theorem gives no additional information as in the case of strong coupling branch.

We then conclude that in the case of exponential dilaton potential of which the asymptotic geometry of the universe is not de Sitter but Minkowski, we find an appropriate C-function defined in the stringy frame for which the holographic C-theorem holds as long as the stringy Hubble constant is positive. For string coupling evolves toward the weak regime the holographic picture defined above is valid only for the accelerating universes in the Einstein frame.

In the above discussion we have assumed the existence of an attractive fixed point in the solution space such that the ratio p/ρ(or ρ/V) is constant and the above check for the holographic C-theorem is valid. However, this is not always true if there is no fixed point such that there is a long transient for $\rho/V \to constant$. A special case is for constant potential V so that one can solve

$$\dot{\Phi} \propto e^{-d\lambda} \tag{42}$$

such that $\rho/V = 1 + \dot{\Phi}^2/V(d-1)$ is not a constant. In this case, for the strong coupling branch, the holographic C-theorem still holds and the stringy Hubble constant is positive definite. On the other hand, for the weak coupling branch, the stringy Hubble constant

$$H_s \propto \sqrt{\frac{\rho}{d(d-1)}} - \sqrt{(d-1)(\rho - V)} \,, \tag{43}$$

which can not be always positive definite and can easily become negative in the early universe where $\rho >> V$, however, the string coupling is strong then and the perturbative picture breaks down. To check the C-theorem we find that

$$\dot{H} - \frac{d}{d-1} H \dot{\Phi} = \frac{-\sqrt{\rho - V}}{d-1} (\sqrt{\rho - V} - \sqrt{d\rho}) \,, \tag{44}$$

which is positive in general and the holographic C-theorem is violated. Note that in the late time the above quantity approach to zero as $\rho \to V$, which signals a finite central charge for the UV fixed point, the dual CFT of the de Sitter space. One can also see this in the Einstein frame.

Another example without the attractive fixed point in the solution space is the tracker solution associated with the power-law potential discussed in the last section. The corresponding Hubble constant in the stringy frame is

$$H_s^{(T)} = \frac{e^{-Z/\sqrt{d-1}}}{\sqrt{d-1}} (\sqrt{\frac{\rho_0}{d}} (-Z)^{-\delta/2} - \sqrt{C_0} (-Z)^{-(\delta+2)/2}) \,. \tag{45}$$

Obviously when the 2nd term dominates in the early universe H_s^T becomes negative, however, then the perturbative picture breaks down. On the other hand, in the late time $H_s^{(T)}$ is positive and the the perturbative picture is good. Similarly, for the C-theorem to hold it is easy to see that one should require

$$\frac{-1}{d-1} (-Z)^{-\gamma} + \frac{\gamma}{2\sqrt{d-1}} (-Z)^{-(\gamma+1)} < 0 \,. \tag{46}$$

Again, the inequality is easily violated in the early Universe but holds true in the late time.

Besides the validity of the perturbative string picture in the bulk, the holographic RG flow and the C-function are highly sensitive to the asymptotical geometry which classifies the bulk geometry and the dual CFT. Even though the C-theorem seems being violated in the early universe for the tracker potential, it holds at the late time where the dark energy dominates and the asymptotical geometry is the same as the one for the exponential-law potential. Therefore, the exponential and the tracker cases should share the same holographic picture in the far future. This justifies that the choice of the stringy C-function is also good for the tracker case.

5 Strong Coupling Branch from Its Dual M-theory

As remarked in the last section for some case the string coupling will grow as the Universe evolves so that the perturbative string picture can not be relied on any more. If so, the non-perturbative string modes become light and a M-theory cycle opens up so that the effective theory becomes a weakly coupled (d+2)-dimensional Einstein gravity

$$S_M = -\int d^{d+2}x\sqrt{-\tilde{G}}[\tilde{R} - \sum_{i=1}^{D}(\tilde{\nabla}\sigma_i)^2 + \tilde{V}(\sigma_i)] \ , \tag{47}$$

where the tilde quantities are with respect to the (d+2)-dimensional metric $\tilde{G}_{ab}$. After compactifying on the following metric ansatz

$$\tilde{G}_{ab}dx^a dx^b = e^{\frac{2}{\sqrt{d}}\Phi}dy^2 + e^{\frac{-2}{d-1}(1+\frac{1}{\sqrt{d}})\Phi}\hat{g}_{\mu\nu}dx^\mu dx^\nu \ , \tag{48}$$

the action S_M will reduce to the stringy action (6) with the potentials are related by $U(\Phi,\sigma_i) = e^{\frac{-2}{d-1}(1+\frac{1}{\sqrt{d}})\Phi}\tilde{V}(\sigma_i)$ at the classical level, which leads to the relation between the potential V in the Einstein frame and $\tilde{V}$ as $V(\Phi,\sigma_i) = e^{\frac{-2}{(d-1)\sqrt{d}}\Phi}\tilde{V}(\sigma_i)$. This relation is too restrictive and could be modified by the quantum correction, also by the mixing of the dilaton and the other internal moduli, we will assume the complications in the following discussions such that the potential V is a generic one.

Note that the metric component $\tilde{G}_{yy}$ is increasing as Φ grows, which reflects the open-up of the M-theory cycle as string coupling becomes strong. Then the Hubble constant and the Newton constant in the M-frame with respect to the metric $\tilde{G}_{\mu\nu} = e^{\frac{-2}{d-1}(1+\frac{1}{\sqrt{d}})\Phi}\hat{g}_{\mu\nu} = e^{\frac{-2}{(d-1)\sqrt{d}}\Phi}g_{\mu\nu}$ in terms of the quantities defined in the Einstein frame with respect to the metric $g_{\mu\nu}$ are

$$H_M = g^{\frac{-1}{(d-1)\sqrt{d}}}(H - \frac{1}{(d-1)\sqrt{d}}\,\dot{\Phi}) \ , \tag{49}$$

$$G_M = g^{\frac{-1}{\sqrt{d}}}G'_N \ , \tag{50}$$

where G'_N is the (d+2)-dimensional Newton constant without moduli dependence.

The inverse C-function in the M-frame is

$$H_M^{d-1}G_M = G'_N(H - \frac{1}{(d-1)\sqrt{d}}\,\dot{\Phi})^{d-1} \ . \tag{51}$$

Similar to the case in stringy frame, the inverse C-function in M-frame is again of no g factor despite of the g factor in H_M. As remarked before, this is a good sign for the choice of the C-function, otherwise, the complication of the g factors will make difference between the leading scale behaviors of the C-functions defined in the M-frame and in the Einstein frame. Unlike the stringy frame case, we do not have the test of the infinite future horizon for the M-frame since $H_M > 0$ always as shown below.

Although there is a relative sign in (49) it is straightforward to see that $H_M \geq 0$ where the equality holds only for $V = 0$. This means that the universe in the M-frame is always expanding. Explicitly, by choosing positive $\dot{\Phi}$ we get

$$H_M = g^{\frac{-1}{(d-1)\sqrt{d}}}\left(1 - \sqrt{\frac{1+\omega}{2}}\right)H \ , \tag{52}$$

so that $H_M > 0$ as long as $\omega < 1$. Note that ω is always smaller than or equal to one for non-ghost canonical scalar even it is a function of time. So, the fact of $H_M > 0$ is true also for non-exponential potential. Similarly to the discussion in the stringy frame, the C-theorem in the M-frame holds true always because $H_M > 0$ and $H_M \propto H$.

6 Conclusion

We know little about nonsupersymmetric string theory with a small, positive vacuum energy. There are two lines one may follow to gain more knowledge. One is to follow basic rules available in string theory to study supersymmetry breaking to get as realistic as possible. Another is to follow what cosmology has to teach us, to push the dichotomy faced in string cosmology as far as possible. The latter line is more "phenomenological". Approaches to formulating a holographic dual of an accelerating universe are more in this spirit. Even in such a "phenomenological" approach, one has very few handles. It appears then that the C-function and the corresponding RG flow is one of such handles.

We explored alternative C-functions in a general situation when the fate of the universe is not exactly de Sitter. The C-function in a universe driven to the weak coupling regime sounds more attractive, since the C-theorem results in a genuine constraint on possible potentials for the dilaton. It is certainly desirable to consider even more general situations when there are more than one moduli.

In a four dimensional universe, we do not have a first principle based upon which a general C-function can be constructed. As a working hypothesis, a general C-function ansatz contains only metric and scalar fields and their first derivatives. The first derivatives can be replaced by their corresponding beta functions in the RG language. Then the C-function in turn determines the beta functions in the following fashion

$$\beta^i = G^{ij}\frac{\partial C}{\partial g^j},$$

where g^i stand for metric and scalar fields. However, without knowing G^{ij} it is impossible to deduce the c-function by comparing the above gradient flow equations and equations of motion derived from a classical action.

For the above reason, it is then desirable to make reasonable guesses in various situations and to check the consequences of these guesses, in order to make progress. The work presented in this note may be viewed as a step in this direction.

Acknowledgments ML was supported by a grant of NSC, and by a "Hundred People Project" grant of Academia Sinica and an outstanding young investigator award of NSF of China. FLL was supported by the NSC grant No. NSC89-2112-M-032-002. The authors would like to thank the courtesy of the CTP at NTU, the NCTS, and the CosPA project, Taiwan.

References

[1] S. Perlmutter et al.(Supernova Cosmology Project Collaboration), "Measurements of Omega and Lambda from 42 High-Redshift Supernovae", Ap. J. 517 (1999) 565;
A. G. Riess et al. (Supernova Search Team Collaboration), "Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant," Astron. J. 116, (1998) 1009.

[2] E. Witten, "Quantum Gravity in de Sitter Space", Talk given at the String Conference 2001, Tata Institute, Mumbai, India, January 2001, http://theory.tifr.res.in/strings/ .

[3] S. Hellerman, N. Kaloper, L. Susskind, "String Theory and Quintessence", hep-th/0104180, JHEP **0106** (2001) 003;
W.Fischler, A.Kashani-Poor, R.McNees, S.Paban, "The Acceleration of the Universe, a Challenge for String Theory", hep-th/0104181, JHEP **0107** (2001) 003.

[4] C. Hull, "De Sitter Space in Supergravity and M Theory", hep-th/0109213, JHEP **0111** (2001) 012, and the references therein.

[5] A. Strominger, "The dS/CFT Correspondence", hep-th/0106113;
V. Balasubramanian, J. de Boer and D. Minic, "Mass, Entropy and Holography in Asymptotically de Sitter Spaces", hep-th/0110108.

[6] M. Li, "Matrix Model for De Sitter", hep-th/0106184;
Y. H. Gao, "Symmetries, Matrices, and de Sitter Gravity", hep-th/0107067.

[7] for a partial list, D. Klemm, "Some Aspects of the de Sitter/CFT Correspondence", hep-th/0106247;
S. Nojiri and S. D. Odintsov, "Quantum cosmology, inflationary brane-world creation and dS/CFT correspondence", hep-th/0107134;
T. Shiromizu, D. Ida, T. Torii, "Gravitational energy, dS/CFT correspondence and cosmic no-hair", hep-th/0109057;
U. H. Danielsson, "A black hole hologram in de Sitter space", hep-th/0110265;
R.-G. Cai, "Cardy-Verlinde Formula and Asymptotically de Sitter Spaces", hep-th/0111093.

[8] T. Banks, "Cosmological Breaking of Supersymmetry?" hep-th/0007146

[9] A. Strominger, "Inflation and the dS/CFT Correspondence", hep-th/0110087.

[10] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, "Renormalization group flows from holography supersymmetry and a c-theorem", Adv. Theor. Math. Phys. 3 (1999) 363;
S. S. Gubser, "Non-confromal examples of AdS/CFT", Class. Quant. Grav. 17 (2000) 1081.

[11] F.-L. Lin and Y.-S. Wu, “The c-Functions of Noncommutative Yang-Mills Theory from Holography”, hep-th/0005054.

[12] W. Fischler and L. Susskind, “Holography and Cosmology”, hep-th/9806039.

[13] A.A. Tseytlin and C. Vafa, “Elements of String Cosmology”, hep-th/9109048.

[14] T. Banks and M. Dine, “Dark Energy in Perturbative String Cosmology”, hep-th/0106276.

[15] P. J. Steinhardt, L. Wang, I. Zlatev, “Cosmological Tracking Solutions”, Phys. Rev. **D59**:123504(1999).

[16] B. Ratra, P.J.E. Peebles, “Cosmological Consequences of a Rolling Homogeneous Scalar Field”, Phys. Rev. **D37**:3406(1988).